Lower Bounds in Algebraic Complexity via Symmetry and Homomorphism Polynomials

Prateek Dwivedi
IT University of Copenhagen
Copenhagen, Denmark
prdw@itu.dk

Benedikt Pago
University of Cambridge
Cambridge, United Kingdom
benedikt.pago@cl.cam.ac.uk

Tim Seppelt
IT University of Copenhagen
Copenhagen, Denmark
tise@itu.dk

Abstract

Valiant's conjecture from 1979 asserts that the circuit complexity classes VP and VNP are distinct, meaning that the permanent does not admit polynomial-size algebraic circuits. As it is the case in many branches of complexity theory, the unconditional separation of these complexity classes seems elusive. In stark contrast, the symmetric analogue of Valiant's conjecture has been proven by Dawar and Wilsenach (ICALP 2020): the permanent does not admit symmetric algebraic circuits of polynomial size, while the determinant does. Symmetric algebraic circuits are both a powerful computational model and amenable to proving unconditional lower bounds.

In this paper, we develop a symmetric algebraic complexity theory by introducing symmetric analogues of the complexity classes VP, VBP, and VF called symVP, symVS, and symVF. They comprise polynomials that admit symmetric algebraic circuits, skew circuits, and formulas, respectively, of polynomial orbit size. Having defined these classes, we show unconditionally that

$$\mathsf{symVF} \subsetneq \mathsf{symVS} \subsetneq \mathsf{symVP}.$$

To that end, we characterise the polynomials in symVF and symVS as those that can be written as linear combinations of homomorphism polynomials for patterns of bounded treedepth and pathwidth, respectively. This extends a previous characterisation by Dawar, Pago, and Seppelt (ITCS 2026) of symVP. The separation follows via model-theoretic techniques and the theory of homomorphism indistinguishability.

Although symVS and symVP admit strong lower bounds, we are able to show that these complexity classes are rather powerful: They contain homomorphism polynomials which are VBP- and VP-complete, respectively. Vastly generalising previous results, we give general graph-theoretic criteria for homomorphism polynomials and their linear combinations to be VBP-, VP-, or VNP-complete. These conditional lower bounds drastically enlarge the realm of natural polynomials known to be complete for VNP, VP, or VBP. Under the assumption $\mathsf{VFPT} \neq \mathsf{VW}[1]$, we precisely identify the homomorphism polynomials that lie in VP as those whose patterns have bounded treewidth and thereby resolve an open problem posed by Saurabh (2016).

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CCS Concepts

• **Theory of computation → Algebraic complexity theory**; **Complexity theory and logic**; **Finite Model Theory**.

Keywords

algebraic complexity, symmetric circuit, homomorphism polynomial, graph homomorphism, complexity monotonicity, treewidth, treedepth, pathwidth, counting width, first-order logic with counting quantifiers, homomorphism indistinguishability

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1 Introduction

Algebraic complexity theory aims to classify families of polynomials according to the size of their smallest algebraic circuit representation. An *algebraic circuit* is a directed acyclic graph whose internal vertices represent addition and multiplication gates and whose input gates are labelled with variables or field constants. The circuit computes a polynomial by propagating the computation from the input gates to the output gate through the internal gates. The size of the circuit is defined as the total number of gates and wires it contains. Two canonical examples of polynomials are the determinant $\mathrm{det}_n = \sum_{\pi \in \mathbf{Sym}_n} \mathrm{sgn}(\pi) \cdot \prod_{i \in [n]} x_{i\pi(i)}$, which admits polynomial-size circuits, and the permanent $\mathrm{perm}_n = \sum_{\pi \in \mathbf{Sym}_n} \prod_{i \in [n]} x_{i\pi(i)}$, for which only exponential-size circuits are known. Valiant's conjecture [65], the central open problem in the field, postulates that (perm_n) does not admit circuits of polynomial size—or equivalently that $\mathsf{VP} \neq \mathsf{VNP}$. Here, VP denotes the class of polynomial families $(p_n)_{n \in \mathbb{N}}$ where each p_n has polynomially bounded degree and can be computed by families of algebraic circuits of size polynomial in n. The class VP is contained in VNP, which consists of all polynomial families that can be reduced to the permanent.

Besides VP and VNP, the complexity classes VF and VBP, which are structurally more restrictive and widely believed to be smaller, have been extensively studied. The class VF comprises all polynomials computed by polynomial-size *formulas*, i.e. tree-shaped circuits, which are incapable of reusing previous subcomputations. The class VBP contains the polynomials computed by polynomial-size skew circuits. A circuit is *skew* if all but at most one child of every multiplication gate are input gates. This model is equivalent to algebraic branching programs [46].

The resulting complexity hierarchy is $\mathsf{VF} \subseteq \mathsf{VBP} \subseteq \mathsf{VP} \subseteq \mathsf{VNP}$. While all inclusions are believed to be strict, any unconditional separation appears to be currently out of reach. Facing this impasse, a substantial line of work has focussed on proving lower bounds on restricted circuit models such as monotone or multilinear circuits and formulas [39, 43]. A shortcoming of these results—such as the exponential lower bound [55] for multilinear formulas computing the permanent—is that they apply equally to the determinant, even though the latter belongs to VBP.

To the best of our knowledge, the only restricted circuit model for which the permanent is provably hard while the determinant is still easy is *symmetric* circuits [23]. A similar result is known for a symmetric computation model that does not involve circuits, namely equivariant determinantal representations of permanent and determinant; these, too, exhibit an exponential separation [42]. In this paper, we study the former model, that is, the symmetry restriction of circuits. We introduce symmetric analogues of VF, VBP, and VP and develop a systematic symmetric algebraic complexity theory. Most notably, we unconditionally separate the symmetric versions of VF, VBP and VP.

Curiously, our symmetric circuit model imposes a restriction only on one aspect of algebraic computation, while it relaxes it in others. Thus, the symmetric classes symVF, symVS, symVP that we define below are a priori not subclasses of, but orthogonal to VF, VBP, and VP.

The sense in which they are restricted is, as already said, symmetry on variables: Many polynomials studied in algebraic complexity are symmetric under certain permutations of the variables. Thus, it is natural to demand that the circuits computing these polynomials also possess these symmetries. To be concrete, consider the permanent $\mathrm{perm}_n \in \mathbb{K}[\mathcal{X}_n]$ in variables $\mathcal{X}_n := \{x_{i,j} \mid i, j \in [n]\}$. The group $\mathbf{Sym}_n \times \mathbf{Sym}_n$ acts on $\mathbb{K}[\mathcal{X}_n]$ by mapping variables $x_{i,j}$ to $x_{\pi(i),\sigma(j)}$ for $(\pi, \sigma) \in \mathbf{Sym}_n \times \mathbf{Sym}_n$. It is not hard to see that perm_n is invariant under this action. We call a polynomial[1] $p \in \mathbb{K}[\mathcal{X}_n]$ *matrix-symmetric* if it is invariant under the action of $\mathbf{Sym}_n \times \mathbf{Sym}_n$ [22].

A *matrix-symmetric circuit* for computing a matrix-symmetric polynomial is one where the action of $\mathbf{Sym}_n \times \mathbf{Sym}_n$ on the input variables can be extended to an automorphism of the circuit, that is, a permutation of the gates that preserves wires and non-wires. This model was first considered by Dawar and Wilsenach [23], who showed that the permanent does not admit matrix-symmetric circuits of subexponential size, while the determinant can be computed with polynomial-size symmetric[2] circuits. To be precise, they established a lower bound on the orbit size rather than total circuit size. The *orbit size* of a matrix-symmetric circuit is the size of the largest set of gates that can all be mapped to each other under the action of $\mathbf{Sym}_n \times \mathbf{Sym}_n$, and it is a lower bound for the total size. So, even if symmetric circuits are permitted to be of arbitrary size, they cannot compute the permanent unless they contain exponentially many gates whose subcircuits are mutually symmetric. This motivates our definition of the complexity classes symVP, symVS, and symVF. For precise definitions of symmetric circuits and orbit size, see Section 2.

[1] Throughout, we fix a field $\mathbb{K}$ of characteristic zero.

[2] The determinant is strictly speaking outside the scope of this article because it is not matrix-symmetric but *square-symmetric*, i.e. it is invariant under the simultaneous action of $\mathbf{Sym}_n$ on rows and columns given by $\pi(x_{i,j}) = x_{\pi(i),\pi(j)}$ for $\pi \in \mathbf{Sym}_n$. Nevertheless, our framework can be adapted to square symmetries, see [22].

Definition 1.1 (Symmetric algebraic complexity classes). Let $p_n \in \mathbb{K}[\mathcal{X}_n]$ be a family of matrix-symmetric polynomials.

(1) $(p_n) \in \mathsf{symVP}$ if the p_n admit matrix-symmetric circuits of polynomial orbit size.
(2) $(p_n) \in \mathsf{symVS}$ if the p_n admit matrix-symmetric skew circuits of polynomial orbit size.
(3) $(p_n) \in \mathsf{symVF}$ if the p_n admit matrix-symmetric formulas of polynomial orbit size.

Note that in the non-symmetric model, $\mathsf{VS} = \mathsf{VBP}$, that is, polynomial-size skew circuits and algebraic branching programs have the same expressive power. In the symmetric model, skew circuits offer a more natural notion of symmetry than ABPs, which is why we define the symmetric analogue of VBP in terms of these.

The symmetric complexity classes are quite powerful, as we prove that symVS and symVP contain VBP- and VP-complete polynomials, respectively. Nonetheless, they remain restricted enough to be meaningful: Dawar and Wilsenach [23] proved that the permanent polynomial does not belong to symVP.

We stress that symVP, symVS, and symVF are in general incomparable with VP, VBP, and VF for three reasons: our symmetric classes only contain matrix-symmetric polynomials, they do not impose a degree bound on the polynomials, and the complexity of symmetric circuits is measured in terms of orbit size rather than total size. We justify these design choices by proving natural characterisations of our symmetric classes culminating in the following unconditional separation.

Theorem B. $\mathsf{symVF} \subsetneq \mathsf{symVS} \subsetneq \mathsf{symVP}$.

To obtain Theorem B, we precisely characterise symVF and symVS in terms of so-called *homomorphism polynomials*. This argument is based on the following fundamental observation: The matrix-symmetric polynomials $p \in \mathbb{K}[\mathcal{X}_n]$, such as the permanent, are precisely the finite linear combinations of homomorphism polynomials of bipartite multigraphs [20, Lemma 4].[3] For a bipartite multigraph F with bipartition $V(F) = A \uplus B$, and a number $n \in \mathbb{N}$, the *n-th homomorphism polynomial* of F is

$$\hom_{F,n} := \sum_{h\colon V(F)\to[n]} \prod_{\substack{ab\in E(F)\\ a\in A, b\in B}} x_{h(a),h(b)}\,. \tag{1}$$

If the variables of $\hom_{F,n}$ are instantiated with the $\{0,1\}$-entries of the bi-adjacency matrix of some (n, n)-vertex bipartite graph G, then $\hom_{F,n}$ evaluates to the number of homomorphisms from F to G. This motivates calling F the *pattern graph* of the homomorphism polynomial.

Dawar, Pago, and Seppelt [20, Theorem 1] showed that a family of matrix-symmetric polynomials (p_n) is in symVP if, and only if, (p_n) can be written as linear combinations of homomorphism polynomials for pattern graphs of bounded treewidth. By significantly refining their arguments, we characterise symVF and symVS, which were not considered before.

[3] Similarly, the square-symmetric polynomials, such as the determinant, are precisely those that can be written as linear combinations of suitably defined homomorphism polynomials from directed looped multigraphs [44, Lemma 6.49]. In order not to deal with loops and directed edges, we focus on matrix symmetries.

Theorem C (Characterising symVF). *A family of matrix-symmetric polynomials (p_n) is in* symVF *if, and only if, (p_n) can be written as linear combinations of homomorphism polynomials for patterns of bounded treedepth.*

While the *treewidth* $\mathrm{tw}(F)$ measures how tree-like a graph F is, the *pathwidth* $\mathrm{pw}(F)$ measures how close F is to being a path, see [6, 52]. In turn, the *treedepth* $\mathrm{td}(F)$ measures how star-like the graph F is [51].

Theorem D (Characterising symVS). *A family of matrix-symmetric polynomials (p_n) is in* symVS *if, and only if, (p_n) can be written as linear combinations of homomorphism polynomials for patterns of bounded pathwidth.*

Theorem B follows from Theorems C and D by invoking involved results from finite model theory [7], graph structure theory [64], and the theory of homomorphism indistinguishability [56], see [62]. At first glance, the need for this technical depth is counter-intuitive since treewidth, pathwidth, and treedepth form a strict hierarchy of graph parameters—a fact that may mislead one to conclude that Theorem B is an immediate corollary of Theorems C and D. However, as we elaborate on in Section 3.2, homomorphism polynomials of non-isomorphic pattern graphs are not necessarily linearly independent. Specifically, there exist polynomials that can be written both as a linear combination of homomorphism polynomials for bounded-treewidth patterns and as one for unbounded-treewidth patterns.

This is a blatant difference from the line of work initiated by Curticapean, Dell, and Marx [13] on the fixed-parameter tractability of so-called *graph motif parameters*, i.e. linear combinations of homomorphism counts $\sum_F \alpha_F \hom(F, \star)$. Here, the same graph parameter $\sum_F \alpha_F \hom(F, \star)$ is evaluated on graphs of arbitrary size and this premise is what renders homomorphism counts of non-isomorphic patterns linearly independent [45]. The titular result of [13] asserts that the fixed-parameter tractability of a parameter $\sum_F \alpha_F \hom(F, \star)$ is dictated by the treewidth of the patterns F whose unique coefficients α_F are non-zero. A substantial and beautiful theory [57, 25, 58, 26, 15] has been developed based on this fundamental observation, which strikingly fails in the non-uniform world of algebraic complexity, see Example 3.1.

In the final part of this paper, we address this issue by studying the non-symmetric algebraic complexity of homomorphism polynomials and thereby also leave the symmetry framework from [20] completely. For these polynomials, we prove novel and very general lower bounds, thereby answering an open question raised by Saurabh [60]. Combined with our characterisations of the symmetric algebraic complexity classes, these lower bounds also shed light on the surprising power of the symmetric classes. They imply that symVP and symVS contain a large variety of VP-complete and VBP-complete polynomials, respectively.

In previous works [27, 47, 8, 37, 60], homomorphism polynomials have been identified as a source for natural complete polynomials for VBP, VP, and VNP. However, these publications only studied homomorphism polynomials $(\hom_{F_n,n})$ for specific families of patterns F_n such as $n \times n$ grids or n-leaf complete binary trees. In the context of monotone circuit complexity, homomorphism polynomials have also been studied [41, 2], here however under the even more restrictive assumption that all F_n are the same graph F. Our Theorem E gives general graph-theoretic criteria for a family of patterns F_n to have homomorphism polynomials $(\hom_{F_n,n})$ that are VBP-, VP-, or VNP-complete. Here, by a *p-family* (F_n) of multigraphs, we mean multigraphs satisfying $\|F_n\| := |V(F_n)| + |E(F_n)| \leq q(n)$ for some polynomial q and all $n \in \mathbb{N}$.

Theorem E. *Let $(F_n)_{n\in\mathbb{N}}$ be a p-family of bipartite multigraphs. Let $\epsilon > 0$.*

1. *If $\mathrm{tw}(F_n) \geq n^\epsilon$ for all $n \geq 2$, then $(\hom_{F_n,n})$ is* VNP*-complete.*
2. *If $\mathrm{tw}(F_n) \in O(1)$ and $\mathrm{pw}(F_n) \geq \epsilon \log(n)$ for all $n \geq 2$, then $(\hom_{F_n,n})$ is* VP*-complete.*
3. *If $\mathrm{pw}(F_n) \in O(1)$ and $\mathrm{td}(F_n) \geq \epsilon \log(n)$ for all $n \geq 2$, then $(\hom_{F_n,n})$ is* VBP*-complete.*

In all cases, hardness holds under constant-depth c-reductions over any field of characteristic zero.

In particular, Theorems D and E and [20] imply that the symmetric classes symVS and symVP respectively contain polynomials that are VBP- and VP-complete. Building on techniques from [14, 12], the proof of Theorem E is by devising a chain of algebraic reductions from the complete polynomials of [27, 37] and employing very recent graph-structural results [9, 32, 36]. In Theorem 3.7, we generalise Theorem E to linear combinations of homomorphism polynomials.

Under the assumption VP ≠ VNP, Theorem E does not ascertain for all pattern families (F_n) whether $(\hom_{F_n,n}) \in$ VP. This is due to the fact that families such as $(\hom_{K_{\log(n),\log(n)},n})$ counting homomorphisms from the complete bipartite graphs $K_{\log(n),\log(n)}$ on $\Theta(\log n)$ vertices are unlikely to be VNP-complete as argued in [60, Proposition 3.6.2], see also [54] and [31, Section 4.4]. In order to deal with such pattern families, a potentially stronger complexity-theoretic assumption is required:

Theorem F. *Let $(F_n)_{n\in\mathbb{N}}$ be a p-family of bipartite multigraphs such that $\mathrm{tw}(F_n)$ is non-decreasing. Unless* VFPT = VW[1]*, it holds that $(\hom_{F_n,n}) \in$* VP *if, and only if, $\mathrm{tw}(F_n) \in O(1)$.*

Here, the complexity classes VFPT and VW[1], as introduced by Bläser and Engels [5], are algebraic analogues of the well-studied parametrised complexity classes FPT and #W[1], see [31, 16]. Assuming VFPT ≠ VW[1], Theorem F exhaustively classifies the homomorphism polynomials which are in VP and thus resolves the open problem posed by Saurabh [60, p. 88] to determine the algebraic complexity of homomorphism polynomials $(\hom_{F_n,n})$ of pattern families (F_n) of treewidth $o(n)$.

Furthermore, Theorem F equates the complexity-theoretic hypothesis VFPT ≠ VW[1] with the collapse of the ability of symmetric and non-symmetric algebraic circuits to compute homomorphism polynomials, i.e. $(\hom_{F_n,n}) \in$ VP $\iff$ $(\hom_{F_n,n}) \in$ symVP for all p-families (F_n) of bipartite multigraphs, see Corollary 3.6. In other words, the hypothesis VFPT ≠ VW[1] and the symmetry assumptions on the circuits seem to be surprisingly related in terms of the lower bounds that can be derived from them.

2 Preliminaries

Consider the variable set $\mathcal{X}_n := \{x_{i,j} \mid i, j \in [n]\}$ with the action of the group $\mathbf{Sym}_n \times \mathbf{Sym}_n$ previously described. For any $S \subseteq \mathcal{X}_n$, let

$$\mathbf{Stab}(S) := \left\{\pi \in \mathbf{Sym}_n \times \mathbf{Sym}_n \;\middle|\; \pi(S) = S\right\} \leq \mathbf{Sym}_n \times \mathbf{Sym}_n .$$

The *pointwise stabiliser* is defined as

$$\mathbf{Stab}^\bullet(S) := \left\{\pi \in \mathbf{Sym}_n \times \mathbf{Sym}_n \;\middle|\; \pi(x) = x \text{ for every } x \in S\right\}.$$

Algebraic Circuits and Formulas. An *algebraic circuit* C over a set of variables $\mathcal{X}$ and a field $\mathbb{K}$ is a directed acyclic graph (possibly with multiedges), where each vertex—called a *gate*—is labelled by an element of $\mathcal{X} \cup \mathbb{K} \cup \{+, \times\}$. We write $\lambda(g) \in \mathcal{X} \cup \mathbb{K} \cup \{+, \times\}$ for the label of a gate. The *input gates*, labelled by elements of $\mathcal{X} \cup \mathbb{K}$, have no incoming edges. Internal gates are labelled with $+$ or $\times$. The circuit has a unique *output gate* which has no outgoing edges. The circuit computes a polynomial in $\mathbb{K}[\mathcal{X}]$ by propagating values from the input gates using addition and multiplication operations at the internal gates. Edges are directed according to the flow of computation, that is, from the output of a gate towards the next gate where the value is used. The *size* of a circuit C, denoted $\|C\|$, is the total number of gates and wires counted with multiplicity. We insist throughout that input gates are unique: For every $x \in \mathcal{X} \cup \mathbb{K}$, there is a unique input gate $g \in V(C)$ with $\lambda(g) = x$.

Whenever a permutation group Γ acts on the variables $\mathcal{X}$, we can define what it means for a circuit C over $\mathcal{X}$ to be Γ-symmetric. Let $\mathbf{Aut}(C) \leq \mathbf{Sym}(V(C))$ denote the group of permutations of gates that preserve non-edges, edges with multiplicities, and gate types. Formally, $\sigma \in \mathbf{Sym}(V(C))$ is in $\mathbf{Aut}(C)$ if, and only if, for every internal gate g, $\lambda(\sigma(g)) = \lambda(g)$, and for every pair of gates (g, h), the number and direction of edges between $(\sigma(g), \sigma(h))$ is the same as between (g, h).

We say that $\pi \in \Gamma$ *extends to an automorphism* of C if there exists a $\sigma \in \mathbf{Aut}(C)$ such that $\lambda(\sigma(g)) = \pi(\lambda(g))$ for every input gate g. Here, $\pi(a) = a$ for every $a \in \mathbb{K}$.

Definition 2.1 (Symmetric circuits). Let Γ be a group acting on $\mathcal{X}$. An algebraic circuit C over $\mathcal{X}$ is Γ-*symmetric* if every $\pi \in \Gamma$ on the input gates $\mathcal{X}$ extends to an automorphism of C.

In this paper, we mostly focus on restricted circuit models such as *formulas* and *skew circuits*. In general, a formula is a circuit without multiedges that is a tree. For symmetric formulas, we require treelikeness only on the internal gates because we always identify input gates that are labelled with the same variable or field element.

Definition 2.2 (Symmetric formulas). A *symmetric formula* is a symmetric circuit C without multiedges such that the subgraph of C induced by the internal gates is a tree.

Definition 2.3 (Symmetric skew circuits). An algebraic circuit is *skew* if for every multiplication gate, at most one of its children is an internal (or a non-input) gate. A *symmetric skew circuit* is a symmetric circuit that is skew.

Note that Definition 2.1 does not require that there is a *unique* circuit automorphism σ to which $\pi \in \Gamma$ extends. Often, however, uniqueness of this σ is desirable in order to have a well-defined action of Γ on the entire circuit. We call a Γ-symmetric circuit C *rigid* if the only circuit automorphism in $\mathbf{Aut}(C)$ that fixes every input gate is the identity. This is equivalent to saying that every $\pi \in \Gamma$ extends to a unique $\sigma \in \mathbf{Aut}(C)$. Thankfully, we may always assume that symmetric circuits are rigid, essentially by removing all redundancies in the circuit. The following Lemma 2.4 generalises [1, Lemma 7] and [19, Lemma 2.3] by the final two assertions.

Lemma 2.4 (Rigidification). *Let Γ be a group acting on a variable set $\mathcal{X}$, and let $\mathbb{K}$ be a field. Let C be a Γ-symmetric circuit over $\mathcal{X} \cup \mathbb{K}$. There exists a Γ-symmetric rigid circuit C' with $\|C'\| \leq \|C\|$ that computes the same polynomial as C. If C is a symmetric formula, then C' is a symmetric formula with multiedges. If C is a symmetric skew circuit, then so is C'.*

Note that rigidifying a symmetric formula with this lemma yields a formula with multiedges, which is not allowed by definition; however, these will only appear at intermediate steps inside proofs, so unless explicitly stated, formulas do not have multiedges.

Complexity Measures and Supports. Let C be a $\mathbf{Sym}_n \times \mathbf{Sym}_n$-symmetric circuit over variables $\mathcal{X}_n$ and a field $\mathbb{K}$ as defined in Definition 2.1. It is standard in algebraic complexity to measure the complexity of C by its size $\|C\|$. However, in the context of symmetric circuits, we also want to take the symmetry group into account. If C is rigid, then for a gate g in C and $(\pi_1, \pi_2) \in \mathbf{Sym}_n \times \mathbf{Sym}_n$, we write $(\pi_1, \pi_2)(g)$ to denote the image of g under the unique automorphism of C that extends (π_1, π_2). The *orbit size* of C is defined as

$$\mathrm{maxOrb}(C) := \max_{g \in V(C)} \left|\left\{(\pi_1, \pi_2)(g) \;\middle|\; (\pi_1, \pi_2) \in \mathbf{Sym}_n \times \mathbf{Sym}_n\right\}\right|.$$

Following previous works, our result relies on the notion of the *support* of a gate, see [1, 18]. Since C is $\mathbf{Sym}_n \times \mathbf{Sym}_n$-symmetric, the polynomial computed at the output gate is invariant under that group action. However, this invariance need not hold for internal gates: The polynomials they compute are generally only invariant under a subgroup of $\mathbf{Sym}_n \times \mathbf{Sym}_n$, namely the stabiliser of the gate. The *support set* of a gate g captures precisely those input variables that remain fixed under such permutations. We denote by $\sup(g) \subseteq [n] \uplus [n]$ the *minimal support* of the stabiliser group $\mathbf{Stab}(g) := \left\{(\pi_1, \pi_2) \in \mathbf{Sym}_n \times \mathbf{Sym}_n \;\middle|\; (\pi_1, \pi_2)(g) = g\right\}$, that is, $\sup(g)$ is the smallest set $S \subseteq [n] \uplus [n]$ for which $\mathbf{Stab}^\bullet(S)$ is a subgroup of $\mathbf{Stab}(g)$. It is known [19, Lemma 2.1] that this smallest set S is indeed unique, as long as there exists at least one $S \subseteq [n] \uplus [n]$ whose intersection with each copy of $[n]$ has size less than $n/2$ and which satisfies $\mathbf{Stab}^\bullet(S) \leq \mathbf{Stab}(g)$. The *maximum support size* of a $\mathbf{Sym}_n \times \mathbf{Sym}_n$-symmetric circuit C is defined as

$$\mathrm{maxSup}(C) := \max_{g \in V(C)} |\sup(g)|.$$

We have the following relationships between orbit and support size. For an intuitive and oversimplified explanation of the lemma, one may imagine that the orbit of a gate g essentially corresponds to the orbit of $\sup(g)$, which has size $\binom{n}{|\sup(g)|}$.

Lemma 2.5 ([19, Lemmas 2.4 and 2.5]). *For each $n \in \mathbb{N}$, let C_n be a $\mathbf{Sym}_n \times \mathbf{Sym}_n$-symmetric rigid circuit. Then $\mathrm{maxOrb}(C_n)$ is polynomially bounded in n if, and only if, there exists a constant $k \in \mathbb{N}$ such that $\mathrm{maxSup}(C_n) \leq k$ for all $n \in \mathbb{N}$.*

3 Main Contributions

For full proofs, see the arXiv version [29].

3.1 Characterisations of symVF and symVS by Homomorphism Polynomials

We give an overview of the proofs of Theorem C and Theorem D, which characterise symVF and symVS as the classes of polynomials that can be expressed as linear combinations of bounded treedepth and pathwidth homomorphism polynomials, respectively. Let us begin with symVF.

Theorem C (Characterising symVF). *A family of matrix-symmetric polynomials (p_n) is in* symVF *if, and only if, (p_n) can be written as linear combinations of homomorphism polynomials for patterns of bounded treedepth.*

The harder direction is to show that any symmetric formula with polynomial orbit size computes a linear combination of homomorphism polynomials of bounded treedepth. Dawar, Pago, and Seppelt [20, Theorem 1] proved that every matrix-symmetric circuit of polynomial orbit size computes a linear combination of homomorphism polynomials of bounded treewidth. Their proof is by induction on the circuit structure and essentially shows that the circuit corresponds to a "simultaneous tree decomposition" of all pattern graphs in the linear combination. This result already implies that the output of a symmetric *formula* of polynomial orbit size must also lie in the class of bounded-treewidth homomorphism counts. It remains to prove that in addition to the treewidth, also the *treedepth* of the pattern graphs is bounded by a constant. The crucial technical insight in [20] is that the support size of the gates corresponds to the treewidth of the pattern graphs. And, by Lemma 2.5, the support size must be constant if the orbit size of the circuit is polynomial.

Here, we refine this argument by taking treedepth into account. The proof consists of two parts. Firstly, we introduce the measure of *support depth* for symmetric circuits. The support depth describes how often the support of a gate changes along any path from a leaf to the root. That is, there exists some element in the support of a child gate h that no longer exists in the support of its parent g, i.e. $\mathrm{sup}(h) \setminus \mathrm{sup}(g) \neq \emptyset$. One may think of this as analogous to forget-nodes in tree decompositions [6].

We show that a symmetric formula with orbit size $O(n^d)$ has support depth at most d. This is the case because, whenever the support changes at some gate g, in the above sense, then the gate g has at least $\Omega(n)$ many mutually symmetric children. In a formula, the subcircuits rooted at these children are all disjoint, so if there were more than d such gates g along any root-to-leaf path, then the orbit size of the formula would be at least $\Omega(n^{d+1})$. This shows that, for symmetric formulas, bounded orbit size implies bounded support depth.

As a next step, we show that when support depth is taken into account, then the technical core of [20] actually yields an upper bound not only on the treewidth but also on the treedepth of the pattern graphs whose homomorphism polynomials are computed by the symmetric formula. It turns out that the support depth translates into a corresponding notion of depth of the tree decomposition [30], which is tied to the treedepth of the pattern graphs.

The easier direction of Theorem C requires showing that whenever F is a bipartite multigraph of treedepth d, then $\mathrm{hom}_{F,n}$ is expressible by a matrix-symmetric formula of orbit size at most $O(n^d)$. The idea is to use the standard dynamic programming approach for inductively computing homomorphism counts along a depth-d elimination tree of F. An *elimination tree* of F is a tree T with the same vertex set as F such that any two vertices that are adjacent in F are in ancestor-descendant relation in T. The proof of [41, Theorem 11] contains an explicit construction of a formula for $\mathrm{hom}_{F,n}$ from such an elimination tree, and we observe that this is symmetric. Any linear combination of homomorphism polynomials of bounded treedepth can then simply be written as a symmetric formula that sums up the individual homomorphism polynomials. This proves the easier direction of Theorem C.

A similar dynamic programming approach also applies in the case of bounded pathwidth: whenever F is a bipartite multigraph of pathwidth d, then $\mathrm{hom}_{F,n}$ can be expressed as a matrix-symmetric skew circuit of orbit size at most $O(n^d)$. This proves the easier direction of our second main result.

Theorem D (Characterising symVS). *A family of matrix-symmetric polynomials (p_n) is in* symVS *if, and only if, (p_n) can be written as linear combinations of homomorphism polynomials for patterns of bounded pathwidth.*

As above, the starting point for the harder direction is to observe that, because the orbit size of the circuits is assumed to be polynomially bounded, the supports of the gates are of some constant size k, implying that the pattern graphs admit constant-width tree decompositions. In the special case of skew circuits, we can show that these tree decompositions can be turned into path decompositions: Intuitively, since every product gate has at most one internal child and the other children are input gates, the tree decomposition produced in this fashion never needs to branch.

3.2 Unconditional Separations of Symmetric Classes

In this section, we show our main result.

Theorem B. symVF $\subsetneq$ symVS $\subsetneq$ symVP.

Given the characterisation of symVS and symVF in Theorems C and D as linear combinations of homomorphism polynomials of bounded treedepth and pathwidth, respectively, it may seem that Theorem B is immediate. For example, the family $(P_n)_{n\in\mathbb{N}}$ of n-vertex paths has bounded pathwidth but unbounded treedepth. On the one hand, Theorem C places $(\mathrm{hom}_{P_n,n})$ in symVS. On the other hand, Theorem D asserts that $(\mathrm{hom}_{P_n,n}) \in$ symVF if, and only if, for every $n \in \mathbb{N}$, $\mathrm{hom}_{P_n,n}$ can be written as a linear combination of homomorphism polynomials of bounded treedepth. Readers familiar with homomorphism counts, e.g. with [13], may recall a result of Lovász [45] asserting that the homomorphism counting functions $\mathrm{hom}(F_i, \star)$ are linearly independent for non-isomorphic graphs F_i. So doesn't this readily imply that $(\mathrm{hom}_{P_n,n}) \notin$ symVF?

It does not. The crux here is that linear independence only holds when the $\mathrm{hom}(F, \star)$ can be evaluated at arbitrarily large graphs. In the non-uniform world of algebraic complexity, homomorphism polynomials can only be evaluated at graphs with a certain number of vertices. Thereby, homomorphism polynomials for non-isomorphic patterns fail to be linearly independent in general. The following example shows that $(\prod_{v,w\in[n]} x_{v,w}) \in$ symVF can be written as linear combinations of homomorphism polynomials of patterns of unbounded treewidth, seemingly placing $(\prod_{v,w\in[n]} x_{v,w})$ outside of symVP.

Example 3.1. For $n \in \mathbb{N}$, the polynomial $\prod_{v,w\in[n]} x_{v,w}$ can be written as

(1) a linear combination of homomorphism polynomials of patterns of bounded treedepth, by Theorem C, since the family $(\prod_{v,w\in[n]} x_{v,w})$ is in symVF, and as
(2) a linear combination of homomorphism polynomials comprising all patterns that are quotients of the complete bipartite graph $K_{n,n}$. This linear combination is obtained by observing that $\prod_{v,w\in[n]} x_{v,w}$ is the subgraph polynomial for the pattern $K_{n,n}$ and applying Möbius inversion [20, Theorem 10]. In particular, $\mathrm{tw}(K_{n,n}) \in \Theta(n)$.

In order to deal with the lack of linear independence, we prove a semantic separation of the functions represented by polynomials in symVF, symVS, and symVP. A similar approach was taken by Dawar and Wilsenach [23] when showing that $(\mathrm{perm}_n) \notin$ symVP: They proved that all polynomials in symVP have counting width $O(1)$ while the permanent has counting width $\Theta(n)$. A family of matrix-symmetric polynomials (p_n) has *counting width* [21] at most $k \in \mathbb{N}$ if, whenever two n-vertex graphs G and H satisfy the same sentences in the k-variable fragment C^k of first-order logic with counting quantifiers, then it holds that $p_n(G) = p_n(H)$. Here it is essential that the matrix symmetries allow the polynomial $p_n \in \mathbb{K}[\mathcal{X}_n]$ to be viewed as a parameter of (n,n)-vertex bipartite graphs. In particular, the value $p_n(G)$ does not depend on the ordering of the rows and columns of the bi-adjacency matrix of G.

The logic C^k is well-studied not only in finite model theory [34] but also e.g. in machine learning [35] due to its connections to the Weisfeiler–Leman algorithm [7, 40] and Graph Neural Networks [66, 50]. Crucially, it was shown in [28, 24] that two graphs G and H satisfy the same C^k-sentences if, and only if, they are *homomorphism indistinguishable* over all graphs of treewidth less than k, i.e. every graph F with $\mathrm{tw}(F) < k$ admits the same number of homomorphisms to G and to H. Homomorphism indistinguishability over various graph classes was shown to characterise diverse graph isomorphism relaxations such as isomorphism [45], quantum isomorphism [48, 38], or equivalence w. r. t. various logics [33, 49, 30, 61]. Beyond that, the distinguishing power of graph isomorphism relaxations can be described using homomorphism indistinguishability [56], see the monograph [62]. We adopt this viewpoint when making the following definition for e.g. $w \in \{\mathrm{tw}, \mathrm{pw}, \mathrm{td}\}$.

Definition 3.2. Let w be an $\mathbb{N}$-valued graph parameter. A family of matrix-symmetric polynomials (p_n) has *w-counting width* at most $k \in \mathbb{N}$ if, whenever two (n,n)-vertex bipartite graphs G and H are homomorphism indistinguishable over all graphs F such that $w(F) < k$, then $p_n(G) = p_n(H)$.

By [28, 24], tw-counting width coincides with the original counting width from [21]. Our Theorems C and D and [20, Theorem 1] yield the following semantic properties of symVP, symVS, and symVF.

Corollary 3.3. *Let (p_n) be a family of matrix-symmetric polynomials.*

(1) *If $(p_n) \in$ symVP, then the tw-counting width of (p_n) is $O(1)$.*
(2) *If $(p_n) \in$ symVS, then the pw-counting width of (p_n) is $O(1)$.*
(3) *If $(p_n) \in$ symVF, then the td-counting width of (p_n) is $O(1)$.*

Towards Theorem B, it remains to show that there are polynomials in symVP and symVS of unbounded pw- and td-counting width, respectively. This is implied by the following theorem, whose first assertion is implied by [20, Theorem 9].

Theorem 3.4. *Let (F_n) be a family of bipartite multigraphs.*

(1) *$(\hom_{F_n,n}) \in$ symVP if, and only if, $\mathrm{tw}(F_n) \in O(1)$.*
(2) *$(\hom_{F_n,n}) \in$ symVS if, and only if, $\mathrm{pw}(F_n) \in O(1)$.*
(3) *$(\hom_{F_n,n}) \in$ symVF if, and only if, $\mathrm{td}(F_n) \in O(1)$.*

In other words, Theorem 3.4 asserts that $\hom_{F_n,n}$ cannot be written as linear combinations of homomorphism polynomials for patterns F'_n with $w(F'_n) \in O(1)$ if $w(F_n)$ is unbounded. To prove the theorem, we show that the w-counting width of $(\hom_{F_n,n})$ is unbounded if $w(F_n)$ is unbounded and apply Corollary 3.3.

Here, the key is to argue that, for $w \in \{\mathrm{tw}, \mathrm{pw}, \mathrm{td}\}$ and $k \in \mathbb{N}$, the class of all graphs F such that $w(F) \leq k$ is *non-uniformly homomorphism distinguishing closed* [20]. A graph class $\mathcal{F}$ is *(uniformly) homomorphism distinguishing closed* [56] if, for all graphs $F \notin \mathcal{F}$, there exist graphs G and H which are homomorphism indistinguishable over $\mathcal{F}$ but admit a different number of homomorphisms from F, see [62, Chapter 6]. It was shown in [53, 62, 30] that the class of all graphs F with $w(F) \leq k$ is homomorphism distinguishing closed, for w and k as above. These results are based on Cai–Fürer–Immerman graphs [7, 56] and duality theorems for the width parameters w [4, 64, 52]. The analogous statement in the non-uniform setting is that the w-counting width of the family $(\hom_{F_n,n})$ is in $O(1)$ iff $w(F_n) \in O(1)$. Hence, if e.g. $(\hom_{F_n,n}) \in$ symVP, then, by Corollary 3.3, the tw-counting width of $(\hom_{F_n,n})$ is in $O(1)$, which implies that $\mathrm{tw}(F_n) \in O(1)$. Theorem B follows from Theorem 3.4 by considering the family P_n of n-vertex paths and the family B_n of n-leaf complete binary trees. It holds that $\mathrm{pw}(P_n) = 1$ and $\mathrm{td}(P_n) \in \Theta(\log n)$ as well as $\mathrm{tw}(B_n) = 1$ and $\mathrm{pw}(B_n) \in \Theta(\log n)$. Hence, $(\hom_{P_n,n}) \in$ symVS $\setminus$ symVF and $(\hom_{B_n,n}) \in$ symVP $\setminus$ symVS.

Theorem 3.4 and Example 3.1 hint at two extremes: While generic matrix-symmetric polynomials may be written as both constant-width and unbounded-width linear combinations of homomorphism polynomials, single homomorphism polynomials do not exhibit this intricacy. Generalising the latter case, we map out a large class of matrix-symmetric polynomials whose symmetric complexity can also be read off from its expansion in terms of homomorphism polynomials: For $n \in \mathbb{N}$, let $p_n = \sum_F \alpha_{F,n} \hom_{F,n}$ be a finite linear combination of homomorphism polynomials for pairwise non-isomorphic bipartite multigraphs F without isolated vertices[4] and coefficients $\alpha_{F,n} \in \mathbb{K}$. The *volume* of such a linear combination is $\mathrm{vol}(n) := \max\{|V(F)| : \alpha_{F,n} \neq 0\}$. For a width parameter w as above, write $\max w(p_n) := \max\{w(F_n) : \alpha_{F,n} \neq 0\}$. Under the assumption that the volume is sublinear, we prove the following Theorem 3.5 by applying the descriptive complexity monotonicity

[4] This assumption does not constitute a loss of generality since coefficients can be rescaled to handle isolated vertices, i.e. $\hom_{F+K_1,n} = n \cdot \hom_{F,n}$. Throughout we consider only hom-linear combinations without isolated vertices.

theorem from [19, Theorem 6.10]. The first assertion is implied by [19, Theorem 6.9].

Theorem 3.5. *Let (p_n) be a family of linear combinations of homomorphism polynomials for patterns without isolated vertices of volume $o(n)$.*

(1) $(p_n) \in \mathsf{symVP}$ *if, and only if,* $\max \operatorname{tw}(p_n) \in O(1)$.
(2) $(p_n) \in \mathsf{symVS}$ *if, and only if,* $\max \operatorname{pw}(p_n) \in O(1)$.
(3) $(p_n) \in \mathsf{symVF}$ *if, and only if,* $\max \operatorname{td}(p_n) \in O(1)$.

To reiterate: For example, Theorem C generally shows that $(p_n) \in \mathsf{symVF}$ if, and only if, each p_n can be written as some linear combination of bounded-treedepth homomorphism polynomials. Theorem 3.5 yields more specifically that for every concrete linear combination $(\sum_F \alpha_{F,n} \hom_{F,n})$ of sublinear volume, the treedepth of the patterns in precisely this linear combination dictates whether it is in symVF. If the volume is linear or superlinear, then this is not true and the linear combination may not be unique, as in Example 3.1.

3.3 Classical Algebraic Lower Bounds for Homomorphism Polynomials

Having identified homomorphism polynomials as the pivotal protagonists of a symmetric algebraic complexity theory, we finally turn to the classical algebraic complexity of homomorphism polynomials. Durand, Mahajan, Malod, de Rugy-Altherre, and Saurabh [27] and Hrubeš [37] presented specific families of patterns (F_n) whose homomorphism polynomials are VBP-, VP-, or VNP-complete. More precisely, they showed that the homomorphism polynomials[5] of the family of paths, of complete binary trees, and of cliques are respectively VBP-, VP-, and VNP-complete. We give a sweeping generalisation of these results by providing graph-theoretic criteria on the (F_n) under which their homomorphism polynomials are VBP-, VP-, or VNP-complete.

Theorem E. *Let $(F_n)_{n\in\mathbb{N}}$ be a p-family of bipartite multigraphs. Let $\epsilon > 0$.*

(1) *If* $\operatorname{tw}(F_n) \geq n^{\epsilon}$ *for all* $n \geq 2$*, then* $(\hom_{F_n,n})$ *is* VNP*-complete.*
(2) *If* $\operatorname{tw}(F_n) \in O(1)$ *and* $\operatorname{pw}(F_n) \geq \epsilon \log(n)$ *for all* $n \geq 2$*, then* $(\hom_{F_n,n})$ *is* VP*-complete.*
(3) *If* $\operatorname{pw}(F_n) \in O(1)$ *and* $\operatorname{td}(F_n) \geq \epsilon \log(n)$ *for all* $n \geq 2$*, then* $(\hom_{F_n,n})$ *is* VBP*-complete.*

In all cases, hardness holds under constant-depth c-reductions over any field of characteristic zero.

Theorem E is proven by reducing from the hard families given in [27, 37]. Our reduction makes use of colourful homomorphism polynomials and techniques developed by Curticapean and Marx [14] and Curticapean [12] for studying #W[1]- and #P-hardness of homomorphism counts of unbounded-treewidth patterns. However, the nature of algebraic computation requires significant alterations to this framework: Most crucially, the patterns F_n are counted only in (n, n)-vertex bipartite graphs. This means that gadget constructions need to be very succinct to fit into the size of the respective target graph. In particular, a literal translation of the techniques from [14, 12] would yield the assertion of Theorem E only for pattern families with volume $|V(F_n)| \leq n^{1-\epsilon}$. To overcome these challenges, we make use of sophisticated graph-structural results [9, 32, 36] and recent insights from homomorphism indistinguishability [56, 63].

Theorem E falls short of establishing the complexity of families of homomorphism polynomials such as $(\hom_{K_{\log(n),\log(n)},n})$ since the treewidth $\Theta(\log(n))$ of the complete bipartite graph $K_{\log(n),\log(n)}$ grows too slowly. In [60, p. 88], Saurabh asked to determine the complexity of homomorphism polynomials $(\hom_{F_n,n})$ for pattern families (F_n) with $\operatorname{tw}(F_n) \in o(n)$. We resolve this question by proving the following theorem:

Theorem F. *Let $(F_n)_{n\in\mathbb{N}}$ be a p-family of bipartite multigraphs such that $\operatorname{tw}(F_n)$ is non-decreasing. Unless* VFPT = VW[1]*, it holds that* $(\hom_{F_n,n}) \in \mathsf{VP}$ *if, and only if,* $\operatorname{tw}(F_n) \in O(1)$.

Here, VFPT and VW[1], as introduced by Bläser and Engels [5], are the algebraic analogues of the well-known parametrised complexity classes FPT and #W[1], see [16]. Both classes comprise parametrised families of polynomials, i.e. families of polynomials $(p_{n,k})$ indexed by both $n, k \in \mathbb{N}$. It holds that $(p_{n,k}) \in \mathsf{VFPT}$ if there exist algebraic circuits for $p_{n,k}$ of size at most $f(k) \cdot q(n)$ for an arbitrary function $f \colon \mathbb{N} \to \mathbb{N}$ and a polynomial q. In contrast, VW[1] contains polynomials that are believed not to be in VFPT.

Note that we do not show that $(\hom_{F_n,n})$ is VW[1]-hard if $\operatorname{tw}(F_n) \in \omega(1)$ as $(\hom_{F_n,n})$ is not even a parametrised family, see [11]. Instead, Theorem F is proven by showing that $(\hom_{F_n,n}) \in \mathsf{VP}$ for $\operatorname{tw}(F_n) \in \omega(1)$ implies the existence of VFPT-circuits for some VW[1]-hard parametrised family of polynomials. Despite this indirect argument, it turns out that the hypothesis VFPT ≠ VW[1] is necessary for proving Theorem F. Furthermore, the following Corollary 3.6 shows that VFPT ≠ VW[1] is equivalent to the collapse of symmetric and non-symmetric computation of homomorphism polynomials, i.e. if $(\hom_{F_n,n}) \in \mathsf{VP}$, then $(\hom_{F_n,n}) \in \mathsf{symVP}$.

Corollary 3.6. *It holds that* VFPT ≠ VW[1] *if, and only if, for every p-family (F_n) of bipartite multigraphs such that $\operatorname{tw}(F_n)$ is non-decreasing,*

$$(\hom_{F_n,n}) \in \mathsf{VP} \Leftrightarrow \operatorname{tw}(F_n) \in O(1) \Leftrightarrow (\hom_{F_n,n}) \in \mathsf{symVP}.$$

Whereas the hypothesis FPT ≠ #W[1] is fundamental to the study of the parametrised complexity of counting problems [16], the hypothesis VFPT ≠ VW[1] has received little attention [11, 3] since it was introduced in 2019. We believe that our Theorem F, representing the algebraic analogue of the seminal characterisation of fixed-parameter tractable homomorphism counts by Dalmau and Jonsson [17], will contribute to the significance of VFPT ≠ VW[1].

Finally, we generalise Theorems E and F to linear combinations of homomorphism polynomials. As in Theorem 3.5, this requires additional growth assumptions. In the following Theorems 3.7 and 3.8, a *p-family* of polynomials (p_n) is one whose degree is bounded by a polynomial in n.

Theorem 3.7. *For $\epsilon > 0$, let $(p_n) := (\sum_F \alpha_{F,n} \hom_{F,n})$ be a p-family of linear combinations of homomorphism polynomials of bipartite multigraphs graphs without isolated vertices of volume at most $n^{1-\epsilon}$ and polynomial* dimension $\dim(n) := |\{F \mid \alpha_{F,n} \neq 0\}|$*. Then the following hold:*

[5] The homomorphism polynomials of [27, 37] are technically different from our Eq. (1) but can be translated.

Table 1: *Overview of the assumptions on matrix-symmetric polynomials represented as linear combinations of homomorphism polynomials subject to which we precisely characterise membership in* VP *and* symVP*. From left to right and from top to bottom, the assumptions become more restrictive.*

	Single homomorphism polynomials ($\hom_{F_n,n}$)	Linear combinations of homomorphism polynomials (p_n)
Symmetric complexity	generic Theorem 3.4	volume $o(n)$ Theorem 3.5
Algebraic complexity assuming VFPT $\neq$ VW[1]	p-family of patterns (F_n) with $\mathrm{tw}(F_n)$ non-decreasing Theorem F	p-family of polynomials (p_n) with volume at most $n^{1-\epsilon}$, polynomial dimension, and non-decreasing $\max \mathrm{tw}(p_n)$ Theorem 3.8
Algebraic complexity assuming VP $\neq$ VNP	p-family of patterns (F_n) with $\mathrm{tw}(F_n) \geq n^\epsilon$ or $\mathrm{tw}(F_n) \in O(1)$ Theorem E	p-family of polynomials (p_n) with volume at most $n^{1-\epsilon}$, polynomial dimension, and $\max \mathrm{tw}(p_n) \geq n^\epsilon$ or $\max \mathrm{tw}(p_n) \in O(1)$ Theorem 3.7

(1) *If* $\max \mathrm{tw}(p_n) \geq n^\epsilon$ *for all* $n \geq 2$*, then* (p_n) *is* VNP*-complete.*
(2) *If* $\max \mathrm{tw}(p_n) \in O(1)$ *and* $\max \mathrm{pw}(p_n) \geq \epsilon \log(n)$ *for all* $n \geq 2$*, then* (p_n) *is* VP*-complete,*
(3) *If* $\max \mathrm{pw}(p_n) \in O(1)$ *and* $\max \mathrm{td}(p_n) \geq \epsilon \log(n)$ *for all* $n \geq 2$*, then* (p_n) *is* VBP*-complete,*

In all cases, hardness holds under constant-depth c-reductions over any field of characteristic zero.

Theorem 3.8. *For* $\epsilon > 0$*, let* (p_n) *be a* p*-family of linear combinations of homomorphism polynomials of bipartite multigraphs without isolated vertices of volume at most* $n^{1-\epsilon}$*, polynomial dimension, and non-decreasing* $\max \mathrm{tw}(p_n)$*. Unless* VFPT = VW[1]*, it holds that* $(p_n) \in$ VP *if, and only if,* $\max \mathrm{tw}(p_n) \in O(1)$*.*

4 Outlook

We have introduced the symmetric algebraic complexity classes symVF, symVS, and symVP as a new perspective on Valiant's programme, and established the unconditional separation symVF $\subsetneq$ symVS $\subsetneq$ symVP via characterisations in terms of homomorphism polynomials. Motivated by this, we also proved new and very general conditional lower bounds for homomorphism polynomials in the classical non-symmetric circuit model, with surprising consequences for the relationship between VP and symVP. See Table 1 for a detailed overview. Further natural questions arising from our work are the following.

Symmetric Depth Reductions. In algebraic complexity theory, natural circuit restrictions such as bounded depth circuits and formulas are extensively studied, in part due to a remarkable depth-reduction phenomenon not known to hold in the Boolean setting. Specifically, any degree-d polynomial computable by a circuit of size s can also be computed by a homogeneous depth-4 circuit or a (possibly non-homogeneous) depth-3 circuit of size $s^{O(\sqrt{d})}$. As a result, proving $n^{\omega(\sqrt{d})}$ lower bounds against such constant-depth circuits would be sufficient to separate VP from VNP and resolve Valiant's conjecture [59]. A natural question is whether such a depth reduction result holds for symmetric circuits as well. Our work indicates that this is unlikely in the general sense. The only kind of depth reduction that may be possible is one where the support depth is not decreased. Namely, Lemma 6.20 in the full version [29] implies that the support depth of a symmetric circuit controls the treedepth of the pattern graphs whose homomorphism polynomials it computes. Thus, Theorems 3.4 and 3.5 represent obstacles to symmetric depth reduction.

Linear Combinations of Linear Volume. Our results show that a central parameter for understanding the complexity of linear combinations ($\sum_F \alpha_{F,n} \hom_{F,n}$) of homomorphism polynomials is their volume $\mathrm{vol}(n) := \max\{|V(F)| : \alpha_{F,n} \neq 0\}$. As summarised in Table 1, linear combinations of sublinear volume behave essentially like single homomorphism polynomials and their symmetric and non-symmetric algebraic complexity is described by our results.

In general, by [20, Lemma 4], every matrix-symmetric polynomial $p \in \mathbb{K}[\mathcal{X}_n]$ can be written as a linear combination of homomorphism polynomials of volume $\leq n$ and such linear combinations are unique by [29, Lemma 8.18]. However, as shown in Example 3.1, it may be possible to represent a polynomial as a linear combination of homomorphism polynomials of smaller width at the expense of increasing the volume to $> n$. This suggests that the realm of linear combinations of at least linear volume is of a fundamentally different nature. Understanding their complexity is an intriguing open problem. Specifically, we ask:

(1) Under which assumptions is the symmetric or non-symmetric algebraic complexity of a linear combination of homomorphism polynomials of linear volume or superpolynomial dimension dictated by the width of the pattern graphs? More concretely, what is the complexity of subgraph polynomials or patterns of linear size such as the permanent, see [20, Conjecture 11]?
(2) Which complexity-theoretic hypothesis does VFPT $\neq$ VW[1] need to be replaced by in order for Corollary 3.6 to hold for pathwidth/VBP/symVS or treedepth/VF/symVF instead of treewidth/VP/symVP, see [10]?
(3) Are there linear combinations (p_n) of homomorphism polynomials of linear volume and polynomial dimension such that $\max \mathrm{tw}(p_n) \in \omega(1)$, but $(p_n) \in$ VP, even if VFPT $\neq$ VW[1]? If not, then symVP and VP would coincide on all matrix-symmetric polynomials, not just on single homomorphism polynomials as shown in Corollary 3.6.

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