On Closure Properties of Read-Once Oblivious Algebraic Branching Programs

Robert Andrews[1], Jules Armand[2], Prateek Dwivedi[3], Magnus Rahbek Dalgaard Hansen[3], Nutan Limaye[3], Srikanth Srinivasan[4], Sébastien Tavenas[2]

[1]University of Waterloo, [2]Université Savoie Mont Blanc, [3]IT University of Copenhagen, [4]University of Copenhagen

Factor Closure It is natural to ask if a computational model is robust under fundamental operations. In the algebraic world, we consider addition, multiplication, and crucially, factoring.

Definition: Algebraic complexity class $\mathscr{C}$ is closed under factoring if for any $f \in \mathscr{C}$, all its irreducible factors f_i are also in $\mathscr{C}$.

Known Results: The following classes are known to be closed under factoring.

- VP (Kaltofen 1985), VNP (CKS 2019, BDS 2025)
- VBP (ST 2021)
- VF and Constant Depth circuits (BKRRSS 2025)

Sparse Non-Closure The class of Sparse Polynomials is a notable exception.

$$f = \prod_{i=1}^{n}(x_i^d - 1) = \underbrace{\prod_{i=1}^{n}(x_i - 1)}_{h} \cdot \underbrace{\prod_{i=1}^{n}\sum_{j=0}^{d-1} x_i^j}_{g}$$

Sparsity gap: f has 2^n monomials, but its factor g has d^n monomials.

Question: Are there models within VP that are not closed under factoring?

Hardness Lifting We "lift" sparsity hardness to the more powerful ROABP model.

Gadget: Replace variable x_i with $x_i x_j$ where (i, j) are edges in an Expander Graph G.

Expander Mixing Lemma ensures that any balanced partition of the vertices (variables) has a large induced matching, giving a large rank Nisan Matrix of the factor.

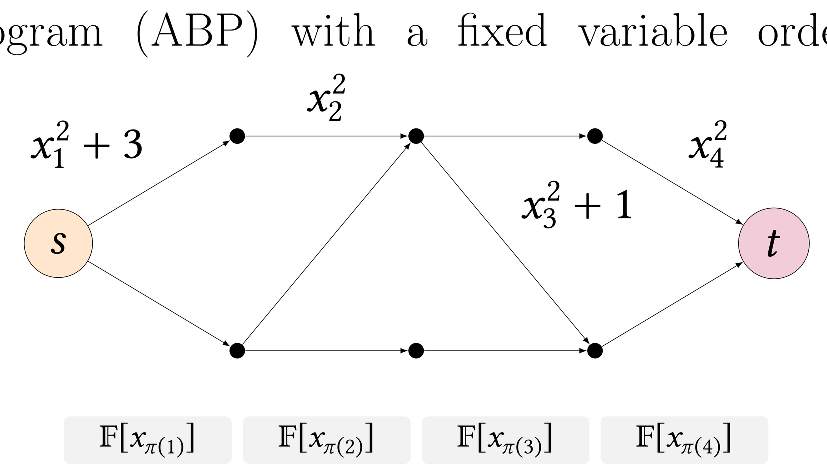
$\Rightarrow$ High sparsity in the base polynomial translates to high rank in the Nisan matrix of the factor.

$\Rightarrow$ High rank Nisan matrix implies lower bounds.

Read-Once Oblivious ABP (ROABP)

A restricted Algebraic Branching Program (ABP) with a fixed variable order π.

$$f = \sum_{\text{path } \gamma : s \rightsquigarrow t} \mathrm{wt}(\gamma).$$



Variable Order Matters: Consider $f = \prod_{i \in [n]}(x_i + y_i)$.

- Good Order: $(x_1, y_1, x_2, y_2, \dots)$ $\Rightarrow$ Width 2.
- Bad Order: $(x_1, \dots, x_n, y_1, \dots, y_n)$ $\Rightarrow$ Width 2^n.

Nisan Matrix: For variable partition $Y \sqcup Z$, index the rows by monomials m_Y and columns by m_Z.

$$M_{Y,Z}(f)[m_Y, m_Z] \quad := \quad \mathrm{coef}_{m_Y \cdot m_Z}(f)$$

Characterization: For any order π and partitions (Y_i, Z_i):

$$\mathrm{Width}(f, \pi) = \max_{i \in [n]} \mathrm{rank}\left(M_{Y_i, Z_i}(f)\right)$$

Factor Non-Closure

Observation (Lifting to Rank): Consider the following polynomial:

$$f \quad = \quad \prod_{i=1}^{n}\left(1 + x_i + \dots + x_i^{d-1}\right).$$

Define $\tilde{f} = f(y_1 z_1, \dots, y_n z_n)$. Then $\mathrm{rank}\left(M_{Y,Z}\left(\tilde{f}\right)\right) \quad = \quad d^n$.

> Theorem (Factor Non-closure): There exists a polynomial $f = g \cdot h$ computable by an ROABP of width $w = 2^{O(n)}$, such that its irreducible factor g requires width $w^{\Omega(\log d)}$ in every variable order.

Proof Sketch: We sketch a weaker version for arbitrary (not necessarily irreducible) factor.

1. Consider an expander graph $G := (V, E)$ with n vertices and constant degree d. Define $P_G = \prod_{(i,j) \in E}(x_i x_j + 1)$.
2. P_G has an ROABP of width $2^{|E|} = 2^{O(n)}$ in the order $(x_1, \dots, x_n)$.
3. $Q_G = \prod_{(i,j) \in E}\left(1 + (x_i x_j) + \dots + (x_i x_j)^{d-1}\right)$. Observe that Q_G is a factor of P_G.
4. Let $\pi : [n] \to [n]$ be any variable order and $Y \sqcup Z$ be the corresponding partition. We show that $\mathrm{rank}(M_{Y,Z}(Q_G)) = d^{\Omega(n)}$.
5. Let M be the induced matching of size $\Omega(n)$ guaranteed by the Expander Induced Matching Lemma. Set the variables not in M to 0, which could only decrease the rank of Nisan's matrix. Then the observation shows that rank is $d^{\Omega(n)}$.

Symmetric Composition

Elementary Symmetric Polynomial (e_d):

$$e_d(x_1, \dots, x_n) \quad := \quad \sum_{1 \le i_1 < \dots < i_d \le n} x_{i_1} \dots x_{i_d}$$

Generating Function:

$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{d=0}^{n} e_d(x_1, \dots, x_n) t^d$$

The Fundamental Theorem: For any symmetric polynomial f_{sym} there is a unique f such that:

$$f_{\mathrm{sym}}(x_1, \dots, x_n) \quad = \quad f(e_1, \dots, e_n)$$

Question: How does the complexity of f relate to f_{sym}?

Previous Results: For circuits, formulas, and constant depth, they are polynomially related (Bläser-Jindal 2019, BKRRSS 2025).

Complexity of Symmetric Composition

We show that for ROABP, the complexity can be exponentially separated.

> Theorem (f_{sym} is easy, f is hard): There exists f such that $f_{\mathrm{sym}} = f(e_1, \dots, e_n)$ has constant-width ROABP, but f requires width $2^{\Omega(n)}$.

Candidate: For any prime k between $n/2$ and n, let $f_{\mathrm{sym}} = \sum_{i=0}^{n} e_i(x_1^k, \dots, x_n^k)$. Then it can be proved that f is the determinant of Circulant matrix.

> Theorem (f is easy, f_{sym} is hard): There exists f with constant-width ROABP such that f_{sym} requires width $2^{\Omega(n)}$.

Candidate: For any $k \le n/2$, let $f(x) = x_k^k$. Then $f_{\mathrm{sym}} = (e_k)^k$.

> Corollary (Powering Non-closure): There exists f with width $O(n)$ such that f^d requires width $2^{\Omega(n)}$ in every variable order.

Open Problems

- Exponential Factor Separation: Is there an easy f for ROABP but some factor g is exponentially harder?
- Hard Roots of Easy Powers: Is there an easy $f = g^d$ for ROABP such that the root g is hard?

References

[1] C. S. Bhargav, Prateek Dwivedi, and Nitin Saxena. A primer on the closure of algebraic complexity classes under factoring. CoRR, abs/2506.19604, 2025.

[2] Somnath Bhattacharjee, Mrinal Kumar, Shanthanu S. Rai, Varun Ramanathan, Ramprasad Saptharishi, and Shubhangi Saraf. Closure under factorization from a result of Furstenberg. CoRR, abs/2506.23214, 2025.